Geometric magnetism in massive chaotic billiards

1997 J. Phys. A: Math. Gen. 30 2853

(http://iopscience.iop.org/0305-4470/30/8/027)

J. Phys. A: Math. Gen. **30** (1997) 2853–2861. Printed in the UK PII: S0305-4470(97)79504-0

Geometric magnetism in massive chaotic billiards

M V Berry and E C Sinclair
H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

Received 12 November 1996

Abstract. Geometric magnetism is a post-adiabatic reaction force exerted by a fast system on a slow system coupled to it. Here it is demonstrated analytically and numerically that a heavy (slow) uncharged billiard boundary, ∂D, swerves, on the average, because of geometric magnetism exerted through elastic impacts from a light (fast) charged particle moving inside ∂D in a magnetic field, for both regular and chaotic fast motions.

1. Introduction

Massive billiards are models for classical coupled fast and slow motions. A planar domain D (figure 1) has an uncharged rigid massive boundary ∂D that can be shifted by elastic impacts from a light particle moving inside it; ∂D has mass M and the light particle has mass $m \ll M$ and charge q. We assume ∂D has a large moment of inertia and so can be translated but not rotated; a way of implementing this is shown in figure 1. There is an external magnetic field $\boldsymbol{B} = B\boldsymbol{e}_Z$ perpendicular to the plane, so that the light particle moves in arcs of Larmor circles. The system has four freedoms, namely the position coordinates $\boldsymbol{r} = \{x, z\}$ of the light mass m, and the position coordinates $\boldsymbol{R} = \{X, Y\}$ of the centre of mass of the heavy boundary M. We shall be particularly interested in billiards for which the light particle moves chaotically for frozen ∂D.

To understand fast/slow systems, it is convenient to make an approximate adiabatic separation of the two time scales. Then the effective dynamics of the slow motion is governed by reaction forces obtained by averaging over the motion of the fast system. For massive billiards $\boldsymbol{r}(t)$ is the fast motion and $\boldsymbol{R}(t)$ the slow motion, and adiabatic averaging requires that m makes many collisions while ∂D hardly moves: defining the fast velocity $\boldsymbol{v} \equiv \dot{\boldsymbol{r}}$ and the slow velocity $\boldsymbol{V} \equiv \dot{\boldsymbol{R}}$, the requirement is $|\boldsymbol{v}| \gg |\boldsymbol{V}|$.

In the most elementary adiabatic averaging (Arnold *et al* 1988, Lochak and Meunier 1988), the reaction force is the gradient with respect to $\boldsymbol{R}$ of the average energy of the fast system for fixed $\boldsymbol{R}$; in quantum mechanics this would be the Born–Oppenheimer force (Messiah, 1962). For the systems we study here, this force is zero because of the translational invariance. Our focus of attention will be on one of the post-adiabatic reaction forces linear in $\boldsymbol{V}$, arising at the next level of approximation, namely *geometric magnetism*.

Obviously, the linear forces can, if local in time and space, be written in the form

$$\boldsymbol{F} = -\mathsf{K}(\boldsymbol{R}) \cdot \boldsymbol{V} \tag{1}$$

familiar from linear response theory, where K is a 2×2 matrix obtained by averaging the fast motion (Berry and Robbins 1993a). Geometric magnetism arises from the antisymmetric part of K, and is the force

$$\boldsymbol{F}_G = \boldsymbol{V} \wedge \boldsymbol{B}_G(\boldsymbol{R}) \tag{2}$$

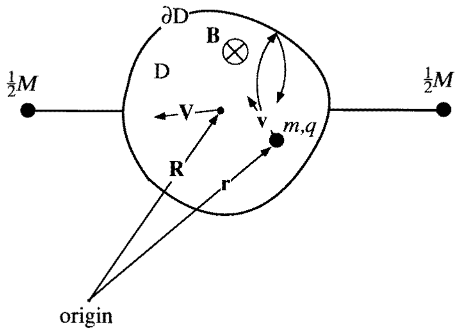

Figure 1. Coordinates and geometry of the massive billiard. The movable part of the boundary ∂D has mass M, centre of mass at $\boldsymbol{R}$ with velocity $\boldsymbol{V}$, and the light particle has mass m, position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$. The long arms provide D with a large moment of inertia, preventing rotation.

where the geometric magnetic field $\boldsymbol{B}_G$ depends on the averaged fast motion. Another interpretation of $\boldsymbol{B}_G$ is that its integral over a surface in $\boldsymbol{R}$ space gives the quantal geometric phase (Berry 1984, Shapere and Wilczek 1989) or (in integrable cases) the classical (Hannay 1985) angle accumulated by the fast system when transported round the boundary ∂S. The reaction force of geometric magnetism is well known in quantum mechanics (Mead and Truhlar 1979, Jackiw 1988, Berry 1989, Cottingham and Hassan 1990, Yin and Mead 1992), and it has been calculated classically for some cases where the fast motion is integrable (Berry and Robbins 1993b, Berry 1996). Here we will demonstrate theoretically (section 2) from elementary arguments and general formulae (Robbins and Berry 1992, Berry and Robbins 1993b), and also computationally (section 3), that it occurs in chaotic classical systems too. We will show that for massive billiards the geometric magnetic field takes the simple form

$$\boldsymbol{B}_G(\boldsymbol{R}) = q\boldsymbol{B} \tag{3}$$

so that the slow system, although uncharged, inherits the magnetism of the fast. This result is a special case of fast motion whose $\boldsymbol{R}$ dependence is obtained by canonical transformation (Robbins 1994).

The symmetric part of K_S of K gives *deterministic friction*, that is a dissipative force

$$\boldsymbol{F}_D = -\mathsf{K}_S(\boldsymbol{R}) \cdot \boldsymbol{V} \tag{4}$$

(Wilkinson 1990, Jarzynski 1992, 1993a, 1995, Berry and Robbins 1993a). In massive billiards there is no deterministic friction, because there is no 'Aristotelian' reference frame with respect to which ∂D can be slowed down.

Geometric magnetism and deterministic friction are not the only post-adiabatic reaction forces. At least three others have been identified: an electric counterpart (Berry 1989, Berry and Robbins 1993b) of geometric magnetism, a velocity-dependent modification of the slow mass (Littlejohn and Weigert 1993), and a memory force (Jarzynski 1993b) depending on the slow history $\boldsymbol{R}(t)$. Beyond these, little is known about the hierarchy of reactions.

2. Theory

If the wall ∂D is represented by the interaction potential $U(\boldsymbol{r}-\boldsymbol{R})$ with the fast particle, the Hamiltonian for the massive billiard is

$$H(\boldsymbol{r},\boldsymbol{p},\boldsymbol{R},\boldsymbol{P}) = \frac{P^2}{2M} + \frac{(\boldsymbol{p}-\frac{1}{2}q\boldsymbol{B}\wedge\boldsymbol{r})^2}{2m} + U(\boldsymbol{r}-\boldsymbol{R}) \tag{5}$$

where $\boldsymbol{p}$ and $\boldsymbol{P}$ are the fast and slow momenta, and we have chosen the symmetric gauge for the magnetic vector potential for the fast particle. We seek the approximate equation of motion for the slow motion $\boldsymbol{R}$, in the adiabatic limit $M \gg m$.

An elementary procedure for calculating $\boldsymbol{B}_G(\boldsymbol{R})$ is based on the observation that the massive billiard has total charge q and the only external force comes from the applied field $\boldsymbol{B}$, so that 'macroscopically', that is as seen from afar, the internal motion would be insignificant and the whole system should move magnetically as a single charged particle with the total mass $M_{\text{tot}} = M+m \approx M$, that is in a Larmor circle. To see how this emerges, we start with the exact equations of motion:

$$\begin{aligned} M\dot{\boldsymbol{V}} &= +\nabla U(\boldsymbol{r}-\boldsymbol{R}) \\ m\dot{\boldsymbol{v}} &= -\nabla U(\boldsymbol{r}-\boldsymbol{R}) + q\boldsymbol{v}\wedge\boldsymbol{B}. \end{aligned} \tag{6}$$

Adding these, we get

$$M_{\text{tot}}\dot{\boldsymbol{V}}_{\text{cm}} = q\boldsymbol{v}\wedge\boldsymbol{B} \tag{7}$$

where $\boldsymbol{V}_{\text{cm}}$ is the velocity of the centre of mass

$$\boldsymbol{R}_{\text{cm}} \equiv \frac{M\boldsymbol{R}+m\boldsymbol{r}}{M+m} \tag{8}$$

of the whole system.

Integrating (7) and taking the cross product with $\boldsymbol{B}$ gives the fast position in terms of $\boldsymbol{V}_{\text{cm}}$ as

$$\boldsymbol{r}-\boldsymbol{C} = -\frac{M_{\text{tot}}\boldsymbol{V}_{\text{cm}}\wedge\boldsymbol{B}}{qB^2}. \tag{9}$$

The vector $\boldsymbol{C}$ is a constant of motion, generalizing the arbitrary centre of the Larmor circle for a single particle in a magnetic field (for a discussion of this invariant, see Avron *et al* (1978)). Hereafter we fix this trivial translational freedom by setting $\boldsymbol{C} = 0$. Then the distance of the light particle from the origin at time t is

$$r(t) = \frac{M_{\text{tot}}V_{\text{cm}}(t)}{qB}. \tag{10}$$

Now we invoke the adiabatic approximation, and assume there exists a time, T_{ad}, for which m collides many times with ∂D while ∂D moves very little. For ergodic fast motion m explores D uniformly. Then, averaging over T_{ad} gives

$$\langle\boldsymbol{r}(t)\rangle \approx \boldsymbol{R}(t) \qquad \langle\boldsymbol{V}_{\text{cm}}(t)\rangle \approx \frac{M\dot{\boldsymbol{R}}(t)+m\langle\dot{\boldsymbol{r}}(t)\rangle}{m+M} = \frac{M}{M_{\text{tot}}}\boldsymbol{V}(t). \tag{11}$$

Thus (9) becomes

$$\boldsymbol{R} \approx -\frac{M\boldsymbol{V}\wedge\boldsymbol{B}}{qB^2}. \tag{12}$$

This is the approximate slow equation of motion. Its solution shows that the boundary moves in a circle with radius

$$R = \frac{MV}{qB} \tag{13}$$

and with constant speed $V = |\boldsymbol{V}|$. The same result could have been obtained by averaging the equation of motion (7) directly. We see that although ∂D is uncharged it inherits magnetism from m. This is the geometric magnetism claimed in (2) and (3), with the Lorentz force involving $\boldsymbol{B}_G = q\boldsymbol{B}$.

Now we carry out the formal exercise of showing that $\boldsymbol{B}_G = q\boldsymbol{B}$ follows from the general formula for geometric magnetism (Berry and Robbins 1993a), valid when the fast motion is chaotic for frozen $\boldsymbol{R}$. The formula is

$$\boldsymbol{B}_G = -\frac{1}{2\partial_E\Omega}\partial_E\left[\partial_e\Omega\int_0^\infty \mathrm{d}t\,\langle(\nabla h)_t\wedge\nabla h\rangle_E\right]. \tag{14}$$

The symbols have the following meaning. h is the Hamiltonian for fast motion with $\boldsymbol{R}$ regarded as a parameter:

$$h(\boldsymbol{r},\boldsymbol{p};\boldsymbol{R}) = \frac{(\boldsymbol{p}-\frac{1}{2}q\boldsymbol{B}\wedge\boldsymbol{r})^2}{2m} + U(\boldsymbol{r}-\boldsymbol{R}) \tag{15}$$

$\partial_E\Omega$ is the weight of the phase-space surface with fast energy E:

$$\partial_E\Omega = \iint \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p}\,\delta(E-h(\boldsymbol{r},\boldsymbol{p};\boldsymbol{R})). \tag{16}$$

Gradients are with respect to $\boldsymbol{R}$. $(f)_t$ denotes the function f on the fast phase space, evaluated at the point $\boldsymbol{r}_t$, $\boldsymbol{p}_t$ that has evolved from $\boldsymbol{r}$, $\boldsymbol{p}$ in time t with frozen $\boldsymbol{R}$. Finally, $\langle f\rangle_E$ is the (microcanonical) phase-space average of the function f over the energy surface E:

$$\langle f\rangle_E \equiv \frac{1}{\partial_E\Omega}\iint \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p}\,f(\boldsymbol{r},\boldsymbol{p})\delta(E-h(\boldsymbol{r},\boldsymbol{p};\boldsymbol{R})). \tag{17}$$

For massive billiards it will be convenient to use the noncanonical but measure-preserving transformation from $(\boldsymbol{r},\boldsymbol{p})$ to $(\boldsymbol{r},m\boldsymbol{v})$. Thus

$$\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}p_x\,\mathrm{d}p_y = m\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{v}. \tag{18}$$

Then a short calculation gives

$$\partial_E\Omega = 2\pi m A \tag{19}$$

where A is the area of the domain for which $E > U(\boldsymbol{r}-\boldsymbol{R})$, which for billiards is the area of D (and of course independent of E). From (15),

$$\nabla h = \nabla h(\boldsymbol{r};\boldsymbol{R}) = -\nabla U(\boldsymbol{r}-\boldsymbol{R}) \tag{20}$$

and so, from (6),

$$(\nabla h)_t = m\ddot{\boldsymbol{r}} - q\dot{\boldsymbol{r}}\wedge\boldsymbol{B} = \partial_t(m\boldsymbol{v}-q\boldsymbol{r}\wedge\boldsymbol{B}). \tag{21}$$

Using this, the time integral in (14), over the frozen motion, can be evaluated, with the result

$$\begin{aligned}\boldsymbol{B}_G &= -\frac{1}{4\pi m A}\partial_E[\partial_E\Omega\langle(m\boldsymbol{v}-q\boldsymbol{r}\wedge\boldsymbol{B})\wedge\nabla U(\boldsymbol{r}-\boldsymbol{R})\rangle_E]\\ &= q\boldsymbol{B}\frac{1}{4\pi m A}\partial_E[\partial_E\Omega\langle\boldsymbol{r}\cdot\nabla U(\boldsymbol{r}-\boldsymbol{R})\rangle_E].\end{aligned} \tag{22}$$

In writing the first equation, the contribution from the upper limit, $t=\infty$, has been ignored (it is elminated by the slightest regularization, for example $\exp(-\varepsilon t)$ in the integrand), and in writing the second equation use has been made of (18) and $\langle\boldsymbol{v}\rangle = 0$ for frozen $\boldsymbol{R}$.

The remaining average in (22) gives

$$\begin{aligned} \boldsymbol{B}_G &= q\boldsymbol{B}\frac{1}{2A}\partial_E \iint_{\mathrm{D}} \mathrm{d}^2\boldsymbol{r}\,\Theta(E-U(\boldsymbol{r}-\boldsymbol{R}))\boldsymbol{r}\cdot\nabla U(\boldsymbol{r}-\boldsymbol{R}) \\ &= q\boldsymbol{B}\frac{1}{2A}\iint_{\mathrm{D}} \mathrm{d}^2\boldsymbol{x}\,\delta(E-U(\boldsymbol{x}))(\boldsymbol{R}+\boldsymbol{x})\cdot\nabla U(\boldsymbol{x}) \\ &= -q\boldsymbol{B}\frac{1}{2A}\iint_{D} \mathrm{d}^2\boldsymbol{x}\,(\boldsymbol{R}+\boldsymbol{x})\cdot\nabla\Theta(E-U(\boldsymbol{x})). \end{aligned} \tag{23}$$

Now

$$(\boldsymbol{R}+\boldsymbol{x})\cdot\nabla\Theta = \nabla\cdot[(\boldsymbol{R}+\boldsymbol{x})\Theta] - \Theta\nabla\cdot(\boldsymbol{R}+\boldsymbol{x}) = \nabla\cdot[(\boldsymbol{R}+\boldsymbol{x})\Theta] - 2\Theta. \tag{24}$$

The divergence gives zero in (23), and the remaining integral gives A and so $\boldsymbol{B}_G = q\boldsymbol{B}$ as claimed in (3).

Although this derivation has made use of the microcanonical distribution for chaotic systems, a similar formulation can be given for integrable fast motion; the result is the same. One way to see this is to adapt the elementary argument leading to (13). For integrable motion the first equation in (11) need not be true, but nevertheless the average $\langle \boldsymbol{r}(t)\rangle$ will differ from $\boldsymbol{R}(t)$ only by a constant shift, and this is sufficient to justify the last equation in (11), leading again to (13).

3. Numerical illustrations

The *circle magnetic billiard* is integrable for frozen $\boldsymbol{R}$ (see figure 2 and Robnik and Berry (1985)). For the general massive billiard, where $\boldsymbol{R}$ is allowed to move, there is an additional

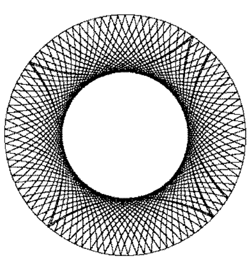

Figure 2. Trajectory of a fast particle inside a frozen circular billiard. The caustic (envelope of the trajectory) indicates integrability. The Larmor radius in the applied magnetic field B is 0.65 times the diameter of the circle; 200 collisions are shown.

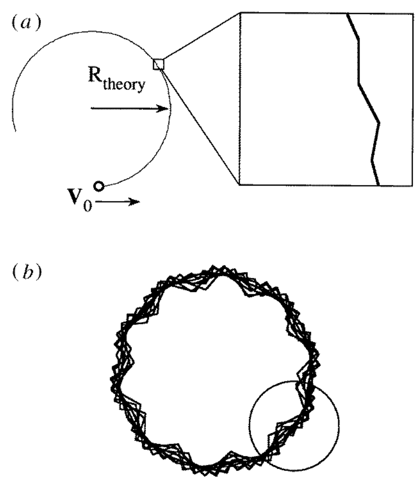

Figure 3. Geometric magnetism for the trajectory of the centre of the massive circular billiard (open circle) with mass ratio (*a*) $m/M = 0.01$ and (*b*) $m/M = 0.4$, initial speed ratio $|V_0|/|v_0| = 0.1$ and initial Larmor radius equal to the diameter of the circle; 600 collisions are shown. In (*a*) the magnification shows the straight segments between collisions.

constant of motion, namely the total angular momentum (which is perpendicular to D and so effectively a scalar)

$$T = \boldsymbol{r} \wedge \boldsymbol{p} + \boldsymbol{R} \wedge \boldsymbol{P}. \tag{25}$$

Together with H and the two components of $\boldsymbol{C}$ defined by (9), this gives four conserved quantities, strongly suggesting (as does numerical evidence to follow) that the massive billiard is integrable. Integrability would hold similarly for any circular potential $U(|\boldsymbol{r}-\boldsymbol{R}|)$, not just billiards. However, these four constants are not involution (they do not commute), so this argument is not watertight. Three mutually commuting quantities are H, T and $\boldsymbol{C}\cdot\boldsymbol{C}$, but we are unable to identify the fourth.

Figure 3(*a*) shows a trajectory of the centre of the circle. It consists of straight segments between impacts from the fast light particle moving inside it. Geometric magnetism is obvious in the 'macroscopic' view of many segments. The theoretical radius from $\boldsymbol{B}_G$, namely MV_0/qB, differs slightly from that observed; the discrepancy can be reduced by the (post-geometric) replacement of M by $M+m$.

Even when m/M is not small, the curvature of the trajectory of the centre of the circle is still apparent, although accompanied by fluctuations. Because the whole system is integrable, these fluctuations form a regular pattern, as figure 3(*b*) shows. The mean radius of the trajectory is about 1.1 times the Larmor radius of the orbit of m (=1.1 circle diameters in this case) which is considerably larger than the 0.25 Larmor radii that geometric magnetism would predict on the basis of equation (13), and the discrepancy is not substantially improved by replacing M by $M+m$. (In the antiadiabatic limit $m \gg M$, mean radius of the centre of the circle would of course be equal to the Larmor radius of the orbit of m.)

The *square magnetic billiard* is nonintegrable for frozen $\boldsymbol{R}$ (Robnik 1986, Berglund and Kunz 1995). There are both regular and chaotic regions in the phase space, with chaos predominating when $|\boldsymbol{B}|$ is such that the Larmor radius is comparable with the length of the side of the square (see figure 4). When $\boldsymbol{R}$ is allowed to move, geometric magnetism is obvious when $m/M \ll 1$ (figure 5(*a*)). When m/M is not small, the curvature of the

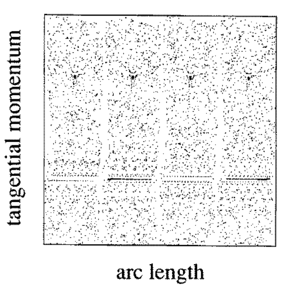

Figure 4. Surface of section for a fast particle inside a frozen square billiard. The Larmor radius in the applied magnetic field, B, is 1.18 times the length of the side of the square; 5000 collisions are shown.

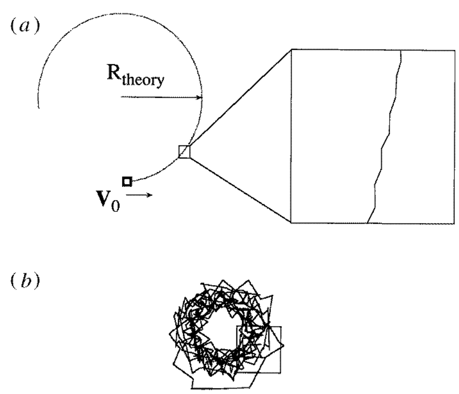

Figure 5. Geometric magnetism for the trajectory of the centre of the massive square billiard (open square) with mass ratio (*a*) $m/M = 0.01$ and (*b*) 0.4, initial speed ratio $|V_0|/|v_0| = 0.1$ and initial Larmor radius equal to 1.18 times the length of the side of the square; 600 collisions are shown. In (*a*) the magnification shows the straight segments between collisions.

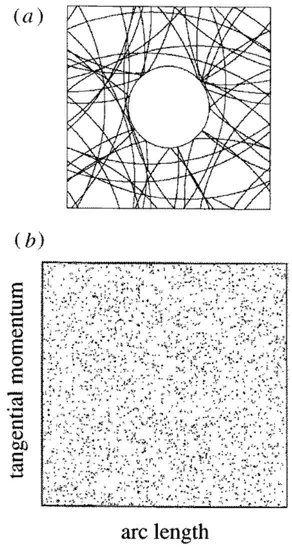

Figure 6. The frozen magnetic Sinai billiard. (*a*) Typical trajectory on the torus, with Larmor radius equal to the box (torus) size, and disc radius 0.28 times the box size. (*b*) Surface of section (registering collisions of the fast particle with the disk), with Larmor radius equal to five times the box (torus) size, and disc radius 0.1 times the box size; 2000 collisions are shown.

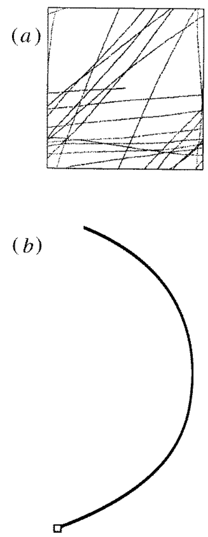

Figure 7. Trajectory of the centre of the disk in the massive magnetic Sinai billiard, with disk radius 0.49 times the box size (so that the disk nearly fills the box), $m/M = 0.001$, and initial speed ratio $|V_0|/|v_0| = 0.1$, (*a*) on the torus (10 000 collisions), where geometric magnetism is present but hard to see, (*b*) on the periodic plane (20 000 collisions), when the torus boundary conditions are unfolded, where geometric magnetism is obvious.

trajectory of the square is still apparent, but the fluctuations are larger than for the circle, and, because of the nonintegrability, irregular, as figure 5(*b*) shows.

In the *Sinai magnetic billiard* ∂D is a disk (with centre $\boldsymbol{R}$) which together with the light (magnetized) particle that strikes it moves not in the plane but in a box with periodic boundary conditions, that is a torus. Therefore the Sinai billiard is inside-out in comparison with the other massive billiards. For frozen $\boldsymbol{R}$, numerical computation (e.g. figure 6) suggests that the motion of the light particle is chaotic for all $|\boldsymbol{B}|$. When $\boldsymbol{R}$ is allowed to move, the trajectory of the disk does not appear to show geometric magnetism when displayed on the torus (figure 7(*a*)), but geometric magnetism is revealed when the trajectory is unfolded into the periodic plane (figure 7(*b*)).

Geometric magnetism is an adiabatic effect, influencing the slow motion on times short enough for nonadiabatic energy exchange between the fast and slow subsystems to be negligible (although long enough for the fast particle to make many collisions). If the fast motion is chaotic, the slow motion over much longer times should be strongly influenced by nonadiabatic effects. We illustrate this by computing the positions of the fast particle for many collisions with ∂D. These should lie within a circle whose radius r_{max} is determined by (10) with the largest value of V_{cm} compatible with the conserved total energy, E, determined by the initial conditions. Since

$$\begin{aligned}(M+m)^2 V_{\mathrm{cm}}^2 &= |M\boldsymbol{V} + m\boldsymbol{v}|^2 \\ &= M^2V^2 + m^2v^2 + 2mM\boldsymbol{V}\cdot\boldsymbol{v} = 2E(M+m) - mM(\boldsymbol{V}-\boldsymbol{v})^2 \qquad (26)\end{aligned}$$

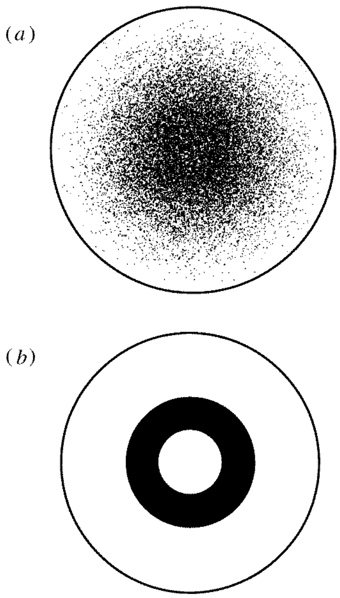

Figure 8. The points mark collisions of the fast particle with ∂D, for (a) the magnetic square billiard with unit side and (b) the magnetic circle billiard with unit radius (here the points fill out the black annulus). The parameters are $m/M = 0.08$, $|V_0|/|v_0| = 0.063$ and the initial Larmor radius is 1, and the large circles show the outer limit $r_{\max} = 3.764$ predicted by (27); 20 000 collisions are shown.

the largest value of $V_{\rm cm}$ occurs when $\boldsymbol{V} = \boldsymbol{v}$, giving

$$r_{\max} = \frac{\sqrt{2EM_{\rm tot}}}{qB} \tag{27}$$

Figure 8(a) shows that for the square massive billiard, whose dynamics is chaotic, the fast particle explores the whole of the interior of this circle. Over such long times it is reasonable to assume equipartition of energy between m and M, and a short calculation then predicts the rms radius of the fast particle to be $r_{\max}/\sqrt{2}$, again in accord with figure 8(a). For the circle massive billiard, which is integrable, the additional constant of motion (25) constrains the particle to an annulus inside $r_{\max}$, as figure 8(b) shows.

Acknowledgment

We thank Dr J M Robbins for helpful comments.

References

Arnold V I, Kozlov V V and Neishtadt A I 1988 *Dynamical systems. III Mathematical aspects of classical and celestial mechanics* (Berlin: Springer)

Avron J E, Herbst I W and Simon B 1978 Separation of centre of mass in homogeneous magnetic fields *Ann. Phys., NY* **114** 431–51

Berglund N and Kunz H 1995 Integrability and erdodicity of classical billiards in magnetic fields *J. Stat. Phys.* **83** 81–126

Berry M V 1984 Quantal phase factors accompanying adiabatic changes *Proc. R. Soc.* A **392** 45–57

——1989 *Geometric Phases in Physics* ed A Shapere and F Wilczek (Singapore: World Scientific) pp 7–28

——1996 The Levitron$^{\rm TM}$: an adiabatic trap for spins *Proc. R. Soc.* A **452** 1207–20

Berry M V and Robbins J M 1993a Chaotic classical and half-classical adiabatic reactions: geometric magnetism and deterministic friction *Proc. R. Soc.* A **442** 659–72

——1993b Classical geometric forces of reaction: an exactly solvable model *Proc. R. Soc.* A **442** 641–58

Cottingham W N and Hassan N 1990 The Berry phase and the hydrogen molecular ion in an external magnetic field *J. Phys. B: At. Mol. Opt. Phys.* **23** 323–7

Hannay J H 1985 Angle variable anholonomy in adiabatic excursion of an integrable Hamiltonian *J. Phys. A: Math. Gen.* **18** 221–30
Jackiw R 1988 *Comm. At. Mol. Phys.* **21** 71–82
Jarzynski C 1992 Diffusion equation for energy in ergodic adiabatic ensembles *Phys. Rev.* A **46** 7498–509
——1993a Energy diffusion in a chaotic adiabatic billiard gas *Phys. Rev.* E **48** 4340–50
——1993b Multiple-time-scale approach to ergodic adiabatic systems: another look *Phys. Rev. Lett.* **71** 839-42
——1995 Thermalization of a Brownian particle via coupling to low-dimensional chaos *Phys. Rev. Lett.* **74** 2937-40
Littlejohn R G and Weigert S 1993 Adiabatic motion of a neutral spinning particle in an inhomogeneous magnetic field *Phys. Rev.* A **48** 924–40
Lochak P and Meunier C 1988 *Multiphase averaging for classical systems* (New York: Springer)
Mead C A and Truhlar D G 1979 On the determination of Born–Oppenheimer nuclear motion wave functions including complications due to conical intersections and identical nuclei *J. Chem. Phys.* **70** 2284–96
Messiah A 1962 *Quantum Mechanics* (Amsterdam: North-Holland)
Robbins J M 1994 The geometric phase for chaotic unitary families *J. Phys. A: Math. Gen.* **27** 1179-89
Robbins J M and Berry M V 1992 The geometric phase for chaotic systems *Proc. R. Soc.* A **436** 631-61
Robnik M 1986 *Nonlinear Phenomena and Chaos* ed S Sarkar (Bristol: Adam Hilger) pp 303–30
Robnik M and Berry M V 1985 Classical billiards in magnetic fields *J. Phys. A: Math. Gen.* **18** 1361–78
Shapere A and Wilczek F eds 1989 *Geometric Phases in Physics* (Singapore: World Scientific)
Wilkinson M 1990 Dissipation by identical oscillators *J. Phys. A: Math. Gen.* **23** 3603–11
Yin L and Mead C A 1992 Magnetic screening of nuclei by electrons as manifestation of geometric vector potential *Theor. Chim. Acta* **82** 397–406